

## Boomerang Fractions

David Wilson suggested this question, which was explored by Gordon Hamilton, Joshua Zucker, Richard Guy, and other participants at the BIRS Conference on Integer Sequences February 27 – March 1, 2015. Bob Klein contributed to this summary.

Choose a fraction  $m/n$ . Begin a sequence of numbers with 1. To get the next number of the sequence, add  $m/n$ . On subsequent steps either add  $m/n$  or take the reciprocal. We say that the *longevity* of the fraction is the minimum number of steps needed to return to 1.

Consider the fraction  $1/2$ . The sequence below shows the quickest way to return to 1 using this fraction, and so the *longevity* of  $1/2$  is 4.

$$1 \rightarrow \frac{3}{2} \rightarrow 2 \rightsquigarrow \frac{1}{2} \rightarrow 1.$$

Investigate some of the questions below or create your own questions about boomerang fractions.

1. What is the longevity of  $1/3$ ? Can you find more than one way to get back to 1 using the minimum amount of steps?
2. What is the longevity of  $1/4$ ?
3. What is the longevity of  $1/5$ ?
4. Can you prove that a fraction of the form  $1/n$  can return to 1?
5. Can you find a bound on the longevity of fractions of the form  $1/n$ ?
6. Are there some fractions for which the boomerang sequence cannot return to 1?
7. Can you prove that a fraction of the form  $\frac{n-1}{n}$  can return to 1?
8. Can you find the longevity of some fractions of the form  $\frac{n-1}{n}$ ?
9. If you make a tree of all of the possible paths for the sequence, how many branches will the tree have of each length?
10. Is it possible to return to 1 if the fraction  $7/9$  is used?