Math Teachers' Circle of Santa Fe
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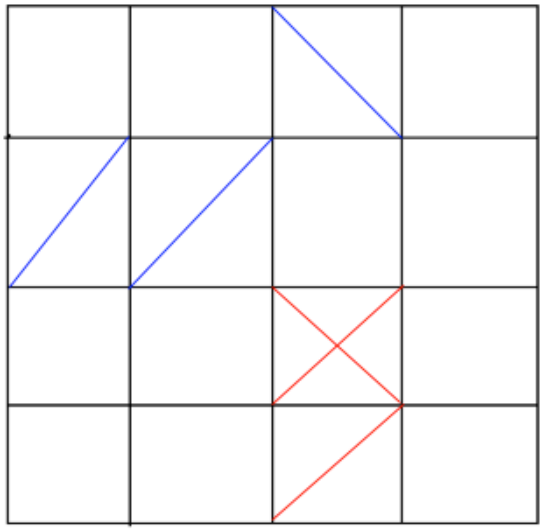
Fun with Diagonals[1]

Sixteen unit squares are arranged to form a square array as shown in the diagram.



Part I

1. What is the maximum number of diagonals that can be drawn in these unit squares so that no two diagonals share a common point (including endpoints)? The RED diagonals fail to satisfy this condition.

(Don't move on to Part II until you have explored Part I a good bit!)

Part II

2. Draw a few (non-crossing) diagonals in the unit squares of a 4×4 grid. Now assume that these diagonals are mirrors, and a ray of light enters the grid, and starts bouncing around. How long can its path be? (Count the length of the path as the number of unit squares you enter. If you enter a unit square twice, then count it twice.)

3. Consider a 4×4 grid again. Color some of the grid points red in such a way that no four red points would form the vertices of a rectangle (with sides parallel to the sides of the grid). What is the maximum number of grid points you can color?

[1] Drawn from Gabriella Pinter's JMM 2017 Demo Math Circle presentation and from the NRICH project and University of Cambridge (http://nrich.maths.org)

Questions and Hints

1. Consider square grids with sides 1,2,3, and 4. What pattern can you see? Does this pattern hold for squares of larger sizes?
2. Hint: There is a nice pattern for square grids with an even side.
3. What about rectangular grids? Can the argument for square grids with an even side be extended to cover the case of some rectangular grids?
4. Can you devise an algorithm that would systematically check, and tell you what the maximum number of diagonals is?
5. What happens to this problem in three dimensions?

References

1. References:OnlineEncyclopediaofIntegerSequences,https://oeis.org/A264041, On the Maximum Number of Non-intersecting Diagonals, J. Integer Sequences, https://cs.uwaterloo.ca/journals/JIS/VOL20/Pinter/pinter3.html
2. Reference: Jim Henle, The Proof and the Pudding: What Mathematicians, Cooks, and You Have in Common, Princeton, 2015; (first chapter available at http://press.princeton.edu/titles/10436.html)
3. Mike Lawler's blog has nice videos on how kids think about this problem:
4. https://mikesmathpage.wordpress.com/2015/07/09/
5. 1988 American Junior High School Math Exam, Problem 16 S. Fenner et.al., Rectangle Free Coloring of Grids,
6. www.cs.umd.edu/~gasarch/papers/grid.pdf