Julia Robinson Mathematics Festival

Directed By Joshua Zucker

The Divisor Game

To play the Divisor Game, choose a number *N* and list all its positive divisors including 1 and *N*. Two players take turns crossing out a number from that list. Each time they cross out a number, they must also cross out all of its divisors. Whoever crosses out *N* is the loser of the game.

Start by playing several games on the boards given for values of *N* up to 10. Note how the divisors are arranged for you to make it relatively easy to play. For larger values of *N* like 360 it may be harder to draw a picture that arranges things so conveniently!

1. For each *N* up to 10, with good strategy, which player should win, the one who goes first or the one who goes second?

2. Add boards drawn in a similar style for *N* = 11 through 20 to your sheet, and figure out what the best strategy is. Should the first or second player be the winner?

3. Who has a winning strategy when *N* is a power of a prime number (such as $2^5 = 32$ or $3^4 = 81$)? Does it depend on the prime, the exponent, both, or neither?

4. Who has a wining strategy when *N* is a product of two different primes (such as 2 3 = 6 or 3 7 = 21 or 7 13 = 91)?

5. Why does it not matter which two primes you picked in the previous problem?

6. To make your answer to the previous problem more precise, a player is said to have a "winning strategy" if the player can guarantee victory before the game starts. Assuming both players are intelligent, rational, and trying to win, prove that one of the two players has a winning strategy. That is, as soon as *N* is chosen, the result of a game between experts is already known, before the game even starts!

7. To generalize your answers to some of the above, explain why the only things that affect who will win the game are the exponents in the prime factorization of *N*; it doesn't matter which primes *N* is made of, only the exponents they are raised to.

8. Which player has a winning strategy when *N* is a product of distinct primes? You already know the answer from previous parts when *N* is formed from 1 or 2 primes, so now you need to look at products of 3 or more primes.

9. How do you determine the winner of the game in general? Can you describe a general pattern, depending on *N*, for which player has a winning strategy?

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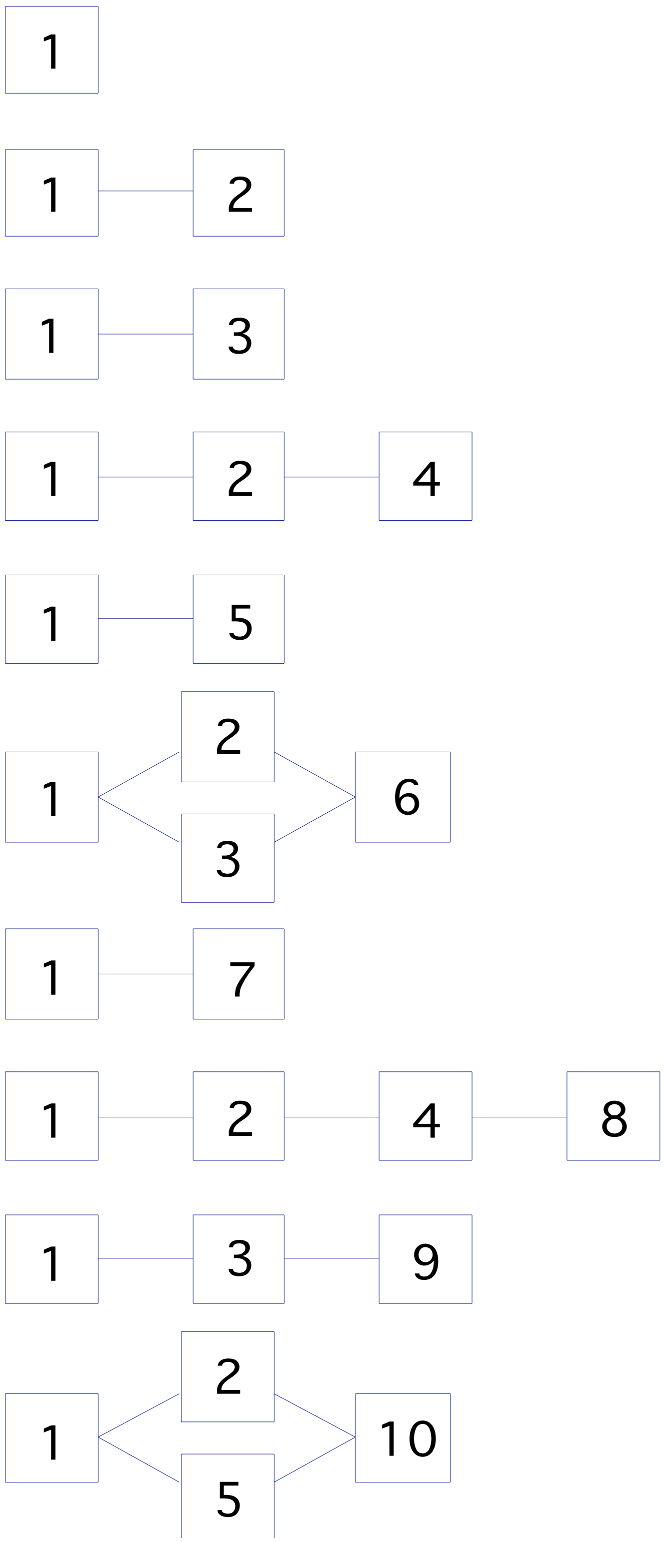
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